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Conserved charges of non-yangian type for the Frahm-Polychronakos spin chain

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Abstract

Through an $\hbar$-expansion of the confined Calogero model with spin exchange interactions, we extract a generating function for the involutive conserved charges of the Frahm-Polychronakos spin chain. The resulting conservation laws possess the spin chain yangian symmetry, although they are not expressible in terms of these yangians.

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1. Introduction

Integrable spin chains with long-range interactions have remarkable properties, not the least being that they furnish a sort of discretization of particular conformal field theories with Lie group symmetry [1, 2, 3]. The archetypal model is the Haldane-Shastry model [4] in which N $su(n)$ spins placed equidistantly on a circle are coupled by a spin-exchange interaction proportional to the inverse square of their chord distance:

$$H^{(\mathrm{HS})} = \frac{1}{2} \sum_{i,j=1}^{N}{}' \frac{z_i z_j}{z_{ij} z_{ji}} P_{ij} \,. \tag{1.1}$$

Here, $z_j \equiv \exp\left(i\frac{2\pi j}{N}\right)$, $z_{ij} \equiv z_i - z_j$ and P_{ij} is the operator which exchanges the i^{th} and j^{th} spins. The primed sum indicates that the summation variables are restricted to differing values.

The Haldane-Shastry model possesses a yangian symmetry algebra which can be taken as a manifestation of its integrability [1]. The conserved charges directly associated to this symmetry are not scalar (they transform in the fundamental representation of $su(n)$) and do not commute among themselves (they generate the yangian algebra, which is non-Abelian). However, from these charges, one can build a set of N scalar commuting operators which turns out to be directly related to those obtained in [5]. However, this set does not explicitly contain the Hamiltonian, contrary to the natural expectation. Moreover, two additional conservation laws were known from brute force calculations [1, 6] but did not appear in this sequence. One expects that, together with $H^{(\mathrm{HS})}$, these represent the first few of a new sequence of a conserved charges. It is natural to try to fit this other sequence in a general scheme based on the fundamental object at the root of integrability : the monodromy matrix. For the Haldane-Shastry model, this has been accomplished by Haldane and Talstra [7]. They showed that the 'new' conservation laws can be obtained by taking a rather subtle limit of the more general dynamical spin model.

For the well-known XXX model, which has short-range interactions, there are also two sets of conservation laws: there is a yangian symmetry [8], out of which scalar conservation laws can be constructed and, in addition, there is a sequence of conservation laws that includes the Hamiltonian [9]. These two types of conservation laws are easily distinguished in models with short-range interactions: the first set is non-local (i.e., the conserved charges involve interactions of all the spins and they become truly non-local in the continuum limit),

while the set containing the Hamiltonian is local (i.e., the n-th member of this sequence has a leading term describing the interaction of n adjacent sites).

For spin chains with long-range interactions, the distinction between locality and non-locality is rather artificial, both sets of charges being manifestly non-local. The difference between these two sets lies in the fact that the Hamiltonian set found by Haldane and Talstra commutes with the symmetry algebra while the yangian set does not. Since both sets commute and can therefore be simultaneously diagonalized, this means that the eigenvalues of the Hamiltonian set are degenerate and characterize a given multiplet while those of the yangian set can be used to label the differing states inside the multiplet.

Let us point out, *en passant*, another major difference between integrable long- and short-range interacting chains, apart from the relativity of the locality concept. For short-range interacting chains, there exists a boost operator that allows for a recursive construction of the local conservation laws. Its origin can actually be traced back to the transfer matrix formalism and the locality of the interaction [10]. No such operator is known for long-range interacting chains.

The argument of [7] relies on a limiting formulation of the Haldane-Shastry spin chain. The model can be viewed as a special reduction of a general Sutherland model (a dynamical model with $\sin^{-2} r$ interaction) with spin degrees of freedom.

The introduction of the spin degrees of freedom in a Calogero-Moser-Sutherland model is rather direct [11] (see also [12]) . If in the classical version of the model, the potential takes the form $\sum g f(r_i, r_j)$ (up to a possible harmonic part), where g is a coupling constant, the quantum version reads $\sum g(g+1) f(r_i, r_j)$. The integrability turns out to be preserved if the term $g(g+1)$ is replaced by $g(g+K_{ij})$ where K_{ij} interchanges the positions i and j. The spin degrees of freedom can be introduced directly by imposing the K_{ij} to be a spin-exchange instead of a position-exchange operator. Another approach, albeit less direct, amounts to retain the position meaning of K_{ij} but consider states that are symmetric under the interchange of both the position and the spin variables. The resulting effect is identical.

The transition from a dynamical model with spin degrees of freedom to the spin chain has been phrased in general terms by Polychronakos in [13]. The idea is simply that from a dynamical model with spin degrees of freedom, we can somehow freeze the latter to generate a spin chain. However, this freezing entails a compatibility condition that follows

from the original equations of motion: the position variables must correspond to the zeroes of the potential. For the $\sin^{-2} r$ interaction potential, this fixes the positions of the chain sites to the roots of unity. Note, on the other hand, that if the potential contains an harmonic piece, this part does not contribute to the spin interaction potential but it enters in the definition of the minima (in fact, whenever it is present, it ensures the existence of these minima).

In this letter, we study the Hamiltonian conservation laws of the Frahm-Polychronakos spin chain [13, 14]. It originates from a Calogero model with inverse square interaction and an harmonic confining potential, augmented with spin degrees of freedom. The potential minima fix the sites of the chain to correspond to the zeroes of the Hermite polynomial $H_N(x)$, to be denoted x_i. The Hamiltonian takes the form

$$H^{(\mathrm{FP})} = \frac{1}{2} {\sum_{i,j=1}^{N}}' \frac{1}{x_{ij}x_{ji}} P_{ij} \tag{1.2}$$

and we will consider the general case of $su(n)$ spins, each of the N spins belonging to the fundamental representation. This model has already been shown to be integrable and to possess a yangian set of commuting operators [13]. In the following, we will show that Haldane-Talstra's argument, formulated here in a somewhat different way, can also be successfully applied to this model, effectively generating the set of Hamiltonian conservation laws.

2. Integrability and Conservation laws

2.1. The yangian algebra $Y[su(n)]$

Let us first briefly review the Yangian algebra $Y[su(n)]$ (see for instance [8, 15]) focusing on its relation to the construction of commuting invariants. For all known integrable spin chains (except, in fact, for a single and somewhat pathological example [16]), the integrability property can be traced back to the existence of a monodromy matrix, an $n \times n$ matrix of operator entries which depends on a spectral parameter u and which satisfies the RTT relation:

$$\mathbf{R}(u-v)\mathbf{T}^{(1)}(u)\mathbf{T}^{(2)}(v) = \mathbf{T}^{(2)}(v)\mathbf{T}^{(1)}(u)\mathbf{R}(u-v)\,. \tag{2.1}$$

Here, the superscripts refer to two auxiliary subspaces in which the matrices act non-trivially, e.g.

$$\mathbf{T}^{(\mathbf{1})}(u) \equiv \mathbf{T}(u) \otimes \mathbf{1}_{\mathbf{n}\times\mathbf{n}} \tag{2.2}$$

and $\mathbf{R}$, called the R-matrix, is an $n^2 \times n^2$ c-number matrix which must satisfy the quantum Yang-Baxter equation:

$$\mathbf{R}^{(\mathbf{12})}(u)\mathbf{R}^{(\mathbf{13})}(u+v)\mathbf{R}^{(\mathbf{23})}(v) = \mathbf{R}^{(\mathbf{23})}(v)\mathbf{R}^{(\mathbf{13})}(u+v)\mathbf{R}^{(\mathbf{12})}(u)\,. \tag{2.3}$$

The RTT relation ensures that the transfer matrix $\mathbf{t}(u)$, which is defined as the trace of the monodromy matrix $\mathbf{t}(u) \equiv \sum_{a=1}^{n} T^{aa}(u)$, satisfies

$$[\mathbf{t}(u), \mathbf{t}(v)] = 0 \tag{2.4}$$

so that its expansion in power series in u^{-1} generates commuting conserved quantities (see below). The Yang-Baxter relation is simply a compatibility relation for the RTT relation.

A simple solution to the Yang-Baxter equation is given by Yang's rational solution

$$\mathbf{R}^{(\mathbf{ij})}(u) = u + \lambda \mathbf{P}^{(\mathbf{ij})}\,, \tag{2.5}$$

where λ is an unspecified deformation parameter and $\mathbf{P}^{(\mathbf{ij})}$ exchanges the auxiliary subspaces i and j

$$\mathbf{P}^{(\mathbf{12})}\mathbf{A}^{(\mathbf{1})}\mathbf{B}^{(\mathbf{2})} = \mathbf{B}^{(\mathbf{1})}\mathbf{A}^{(\mathbf{2})}\mathbf{P}^{(\mathbf{12})}\,. \tag{2.6}$$

With this choice of R-matrix and with the monodromy matrix expanded in a Laurent series as (denoting the ab matrix entry of $\mathbf{T}(u)$ as T^{ab})

$$T^{ab}(u) = \delta^{ab} + \lambda \sum_{m=0}^{\infty} u^{-(m+1)} T^{ab}_m\,, \tag{2.7}$$

the RTT relation reduces to the following commutation relation

$$\left[\, T^{ab}_\ell\,,\, T^{cd}_m \,\right] = \delta^{ad} T^{cb}_{\ell+m} - \delta^{cb} T^{ad}_{\ell+m} + \lambda \sum_{k=0}^{\ell-1} \left\{ T^{cb}_{k+m} T^{ad}_{\ell-k-1} - T^{cb}_{\ell-k-1} T^{ad}_{k+m} \right\}. \tag{2.8}$$

From this structure, we can define two sets of commuting operators. One of these is obtained by the spectral expansion of the transfer matrix

$$\left[\, I_\ell\,,\, I_m \,\right] = 0 \quad , \quad I_m \equiv \sum_{a=1}^{n} T^{aa}_m\,. \tag{2.9}$$

The other set is related to the quantum determinant of the monodromy matrix [17]

$$\mathrm{Det_q}\big[\mathbf{T}(u)\big] \equiv \sum_{\sigma\in S_n} \epsilon(\sigma) T^{1\sigma(1)}\big(u-(n-1)\lambda\big) T^{2\sigma(2)}\big(u-(n-2)\lambda\mathbf{i}gr)\ldots T^{n\sigma(n)}(u)\,. \quad (2.10)$$

Here, $\sigma(i)$ is the image of i under the permutation σ, $\epsilon(\sigma)$ is the permutation's parity and the sum is taken over all permutations of $(1\ldots n)$. The quantum determinant is analogous to the Casimir operator of a Lie algebra in that it commutes with all generators:

$$\Big[\mathrm{Det_q}\big[\mathbf{T}(u)\big]\,,\,\mathbf{T}(v)\Big] = 0\,. \quad (2.11)$$

This property allows one to define a second set of commuting operators from the coefficients of the series expansion of $\mathrm{Det_q}\big[\mathbf{T}(u)\big]$ in terms of the spectral parameter

$$\big[\,J_m\,,\,J_\ell\,\big] = 0 \quad , \quad \mathrm{Det_q}\big[\mathbf{T}(u)\big] \equiv 1 + \sum_{m=0}^{\infty} u^{-(m+1)} J_m\,. \quad (2.12)$$

A given Hamiltonian H will therefore be shown to be integrable if one can prove its symmetry under a non-trivial monodromy matrix

$$\big[\,H\,,\,\mathbf{T}(u)\,\big] = 0\,, \quad (2.13)$$

which guarantees the conserved character of the involutive sets I_m and J_m. However, monodromy matrices are formidable objects which usually do not allow such commutators to be directly carried out. It is therefore very useful to codify the monodromy matrix in a minimal form. Such a minimal coding can sometimes be realized in terms of the yangian algebra $Y[su(n)]$. In fact, defining the lower-order yangian generators (Q_0^{ab}, Q_1^{ab}) $(a,b = 1\ldots n)$ by [2]

$$Q_0^{ab} \equiv -T_0^{ab} \qquad , \qquad Q_1^{ab} \equiv -T_1^{ab} + \frac{\lambda}{2}(\mathbf{T}_0\mathbf{T}_0)^{ab}\,, \quad (2.14)$$

the first few of the commutation relations (2.8) read

$$\begin{aligned}
&\big[\,Q_0^{ab}\,,\,Q_0^{cd}\,\big] = \delta^{bc}Q_0^{ad} - \delta^{da}Q_0^{cb}\\
&\big[\,Q_0^{ab}\,,\,Q_1^{cd}\,\big] = \delta^{bc}Q_1^{ad} - \delta^{da}Q_1^{cb}\\
&\Big[\,Q_0^{ab}\,,\,\big[\,Q_1^{cd}\,,\,Q_1^{ef}\,\big]\Big] - \Big[\,Q_1^{ab}\,,\,\big[\,Q_0^{cd}\,,\,Q_1^{ef}\,\big]\Big] =\\
&\quad \frac{\lambda^2}{4}\bigg\{\Big[\,Q_0^{ab}\,,\,\big[\,(\mathbf{Q}_0\mathbf{Q}_0)^{cd}\,,\,(\mathbf{Q}_0\mathbf{Q}_0)^{ef}\,\big]\Big] - \Big[\,(\mathbf{Q}_0\mathbf{Q}_0)^{ab}\,,\,\big[\,Q_0^{cd}\,,\,(\mathbf{Q}_0\mathbf{Q}_0)^{ef}\,\big]\Big]\bigg\}\,.
\end{aligned} \quad (2.15)$$

[2] Here and hereafter, we use the obvious matrix notation $(\mathbf{T}_0\mathbf{T}_0)^{ab} \equiv \sum_{c=1}^{n} T_0^{ac}T_0^{cb}$

These three relations define, or more precisely, completely characterize the yangian algebra $Y[su(n)]$. The third relation is a sort of compatibility requirement on the different ways to reach Q_2 from multiple commutations involving lower-order charges.

One can reconstruct the whole monodromy matrix strictly from its lower-order yangians whenever the former possesses a trivial quantum determinant

$$\mathrm{Det_q}\big[\mathbf{T}(u)\big] = \text{c-number}\,. \tag{2.16}$$

To justify the last statement, consider the following special cases of the algebra (2.8) :

$$\begin{aligned} T^{ad}_{m+1} &= \big[\, T^{cd}_m \,,\, T^{ac}_1 \,\big] + \lambda (T^{cc}_m T^{ad}_0 - T^{cc}_0 T^{ad}_m) \qquad & (a \neq d) \\ T^{aa}_{m+1} - T^{cc}_{m+1} &= \big[\, T^{ca}_m \,,\, T^{ac}_1 \,\big] + \lambda (T^{cc}_m T^{aa}_0 - T^{cc}_0 T^{aa}_m) \qquad & (\text{no sum})\,. \end{aligned} \tag{2.17}$$

The first of these relations allows us to compute any $T^{ab}_m (a \neq b)$ in terms of the lower-order generators but the second relation is not sufficient to compute the T^{aa}_m by recurrence. However, if (2.16) holds, its spectral expansion gives a set of conditions on $\sum_{a=1}^n T^{aa}_m$ which, when supplemented by (2.17), allow one to compute all the T^{ab}_m from the lower-order yangians. This allows for a tremendous simplification because the symmetry of H under $\mathbf{T}(u)$ then follows from its symmetry under the induced $Y[su(n)]$ representation

$$\text{if} \Big\{ \mathrm{Det_q}\big[\mathbf{T}(u)\big] = \text{c-number} \Big\} \text{then} \Big\{ \big[\, H \,,\, Q^{ab}_{(0,1)} \,\big] = 0 \,\Rightarrow\, \big[\, H \,,\, \mathbf{T}(u) \,\big] = 0 \Big\}. \tag{2.18}$$

When dealing with an irreducible representation of $\mathbf{T}(u)$, the quantum determinant must necessarily be proportional to the identity and the monodromy matrix can then be represented by its lower-order yangians. However, when the considered representation is reducible, one must exercise care because the quantum determinant may then be a non-trivial operator. In the case of the Frahm-Polychronakos model, we will see that the symmetry algebra is reducible but nevertheless possesses a trivial quantum determinant so that in this special case (and for the Haldane-Shastry model), the monodromy matrix will be solely expressed in terms of its *reducible* lower-order yangians.

2.2. The yangian representation in terms of Dunkl operators

Having discussed the theory of $su(n)$ yangians, we now focus on the construction of specific realizations useful for long-range interaction models. First of all, we work in a Hilbert space of N particles endowed with $su(n)$ spin, in which the position (momentum)

operator of particle i will be denoted by R_i (P_i); its spin operators are chosen to be the n^2 fundamental generators E_i^{ab} ($a, b = 1 \dots n$) satisfying

$$\left[E_i^{ab} \, , \, E_j^{cd} \right] = \delta_{ij} \left(\delta^{bc} E_i^{ad} - \delta^{ad} E_i^{cb} \right) \, . \tag{2.19}$$

We now define an hermitian exchange operator $\hat{K}_{ij}$, which permutes the positions of particles i and j

$$\hat{K}_{ij} \mid r_1^{(1)} \dots r_i^{(i)} \dots r_j^{(j)} \dots r_N^{(N)} > \, = \, \mid r_1^{(1)} \dots r_j^{(i)} \dots r_i^{(j)} \dots r_N^{(N)} > \quad . \tag{2.20}$$

We stress that in our notation, operator subscripts refer to particles whereas ket subscripts (superscripts) refer to positions (particles) so that

$$R_i \mid r_1^{(1)} \dots r_p^{(i)} \dots r_N^{(N)} > \, = r_p \mid r_1^{(1)} \dots r_p^{(i)} \dots r_N^{(N)} > \quad . \tag{2.21}$$

The permutation operator $\hat{K}_{ij}$ satisfies

$$\begin{array}{lll} \hat{K}_{ij} f(R_i, P_i) = f(R_j, P_j) \hat{K}_{ij} & \qquad \hat{K}_{ij} \hat{K}_{jk} = \hat{K}_{ik} \hat{K}_{ij} & \\ & & (k, \ell \neq i, j) \; . \\ \hat{K}_{ij} f(R_\ell, P_\ell) = f(R_\ell, P_\ell) \hat{K}_{ij} & \qquad \hat{K}_{ij} \hat{K}_{k\ell} = \hat{K}_{k\ell} \hat{K}_{ij} & \end{array} \tag{2.22}$$

Here, the caret is used to stress that $\hat{K}_{ij}$ is an abstract Hilbert-space operator and therefore acts trivially on any c-number. This contrasts with the K_{ij} operator generally used, which exchanges the position eigenvalues according to $K_{ij} r_i = r_j K_{ij}$ and which is simply the position-space representation of this abstract operator:

$$< r_1^{(1)} \dots r_N^{(N)} \mid \hat{K}_{ij} \mid \psi > \, = K_{ij} < r_1^{(1)} \dots r_N^{(N)} \mid \psi > \quad . \tag{2.23}$$

In a similar way, one can also define a spin exchange operator

$$\begin{array}{ll} P_{ij} E_i^{ab} = E_j^{ab} P_{ij} & \\ & \quad (\ell \neq i, j) \; . \\ P_{ij} E_\ell^{ab} = E_\ell^{ab} P_{ij} & \end{array} \tag{2.24}$$

In the fundamental basis, this operator takes the simple form

$$P_{ij} = \sum_{a,b=1}^{n} E_i^{ab} E_j^{ba} \, . \tag{2.25}$$

Now, in order to eventually establish a link between spatial and spin models, one introduces a projection Π [18], which consists in projecting onto states that are symmetric

with respect to the joint interchange of position and spin variables, that is, states satisfying $\hat{K}_{ij}P_{ij} = 1$. In practice, this projection boils down to the following operation: in a given expression, we move all $\hat{K}_{ij}$ operators to the right and replace them by P_{ij} operators acting in reverse order; e.g.

$$\Pi \; \{\hat{K}_{ij}\hat{K}_{jk}\} = P_{jk}P_{ij} \, . \tag{2.26}$$

This projection possesses the following crucial properties

$$\begin{aligned} \Pi\{AB\} &= \Pi\{A\} \cdot \Pi\{B\} \qquad \text{if} \qquad \big[\, B \, , \, \hat{K}_{ij}P_{ij} \,\big] = 0 \\ \Pi\{AB\} &= \Pi\{B\} \cdot \Pi\{A\} \qquad \text{if} \qquad \big[\, A \, , \, \Pi\{B\} \,\big] = 0 \, . \end{aligned} \tag{2.27}$$

Using this projection technique, one can construct a spin representation of the algebra (2.8) [19]. This representation is based on given position-space Dunkl operators $\bar{D}_i$ (and from now on, we will use the overhead bar to indicate that an operator acts non trivially only in position space) obeying

$$\begin{aligned} &\hat{K}_{ij}\bar{D}_i = \bar{D}_j\hat{K}_{ij} \\ &\big[\, \bar{D}_i \, , \, \bar{D}_j \,\big] = \lambda(\bar{D}_i - \bar{D}_j)\hat{K}_{ij} \, . \end{aligned} \tag{2.28}$$

By induction, one can in fact prove the more general commutation relation

$$\big[\, \bar{D}_i^{\ell} \, , \, \bar{D}_j^{m} \,\big] = \lambda \sum_{k=0}^{\ell-1} \Big(\bar{D}_i^{k+m} \bar{D}_j^{\ell-k-1} - \bar{D}_i^{\ell-k-1} \bar{D}_j^{k+m} \Big) \hat{K}_{ij} \, , \tag{2.29}$$

from which one can immediately define the following involutive set

$$\big[\, \bar{I}_\ell \, , \, \bar{I}_m \,\big] = 0 \quad , \quad \bar{I}_m \equiv \sum_{i=1}^{N} \bar{D}_i^m \, . \tag{2.30}$$

These quantities are purely spatial; in order to define spin invariants, we can use the properties (2.27) and (2.29) to show that the currents

$$T_m^{ab} = \sum_{i=1}^{N} E_i^{ba} \Pi \big\{ \bar{D}_i^m \big\} \tag{2.31}$$

satisfy the monodromy matrix algebra (2.8). The involutive I_m set associated with this algebra is then simply given by

$$\big[\, I_m \, , \, I_\ell \,\big] = 0 \quad , \quad I_m = \Pi \big\{ \bar{I}_m \big\} = \sum_{i=1}^{N} \Pi \big\{ \bar{D}_i^m \big\}. \tag{2.32}$$

Quite remarkably, the monodromy matrix (2.31) can also be expressed in the form $\mathbf{T}(u) = \Pi\{\mathbf{T}'(u)\}$, with $\mathbf{T}'(u)$ another representation of the algebra (2.8), given by [18]

$$\mathbf{T}'(u) = \frac{1}{\bar{\Delta}(u)} \prod_{i=1}^{N} \Big\{ (u - \bar{D}'_i) \mathbf{1}_i + \lambda \mathbf{E}_i^{\top} \Big\} \quad , \quad \bar{\Delta}(u) \equiv \prod_{i=1}^{N} (u - \bar{D}'_i) \tag{2.33}$$

where the spin operators have been grouped in matrix form according to $\big(\mathbf{E}_i^{\top}\big)^{ab} = E_i^{ba}$ and the product over i includes both the usual linear matrix product as well as the tensor one. Here, the $\bar{D}'_i$ are modified Dunkl operators

$$\bar{D}'_i \equiv \bar{D}_i - \lambda \sum_{\substack{j=1 \\ j<i}}^{N} \hat{K}_{ij} \, , \tag{2.34}$$

which satisfy the degenerate affine Hecke algebra with respect to position-space permutations

$$\begin{aligned} &\hat{K}_{ii\pm1} \bar{D}'_i - \bar{D}'_{i\pm1} \hat{K}_{ii\pm1} = \pm\lambda \\ &\big[\, \bar{D}'_i \, , \, \bar{D}'_j \,\big] = 0 \, . \end{aligned} \tag{2.35}$$

Using this Hecke algebra, one can show that

$$\big[\, \bar{\Delta}(u) \, , \, \hat{K}_{ii+1} \,\big] = \big[\, \bar{\Delta}(u) \, , \, \hat{K}_{ii+1} P_{ii+1} \,\big] = 0 \, , \tag{2.36}$$

which can in turn be used to prove that $\big[\, \mathbf{T}'(u) \, , \, \hat{K}_{ii+1} P_{ii+1} \,\big] = 0$. But since any permutation can be expressed as a product of transpositions, we actually have

$$\big[\, \bar{\Delta}(u) \, , \, \hat{K}_{ij} \,\big] = \big[\, \bar{\Delta}(u) \, , \, \hat{K}_{ij} P_{ij} \,\big] = \big[\, \mathbf{T}'(u) \, , \, \hat{K}_{ij} P_{ij} \,\big] = 0 \, . \tag{2.37}$$

Now the quantum determinant of $\mathbf{T}'(u)$ having already been calculated as

$$\mathrm{Det_q}\big[\mathbf{T}'(u)\big] = \frac{\bar{\Delta}(u+\lambda)}{\bar{\Delta}(u)} \, , \tag{2.38}$$

one can use the property (2.37) to factorize the projection and calculate the quantum determinant of $\mathbf{T}(u)$ as [18]

$$\mathrm{Det_q}\big[\mathbf{T}(u)\big] = \mathrm{Det_q}\big[\Pi\{\mathbf{T}'(u)\}\big] = \Pi\Big\{ \mathrm{Det_q}\big[\mathbf{T}'(u)\big] \Big\} = \Pi\left\{ \frac{\bar{\Delta}(u+\lambda)}{\bar{\Delta}(u)} \right\} . \tag{2.39}$$

In principle, one can extract the $\{J_k\}$ set from this formula but the result is highly cumbersome. It is much simpler to focus instead on $\Delta(u)$. Indeed, the relations (2.35) and (2.37) also imply

$$\big[\,\bar{\Delta}(u)\,,\,\bar{\Delta}(v)\,\big] = \big[\,\Delta(u)\,,\,\Delta(v)\,\big] = 0\,, \tag{2.40}$$

where $\Delta(u) \equiv \Pi\{\bar{\Delta}(u)\}$. One can therefore define a simple set of commuting operators by using $\frac{\partial}{\partial u}\ln[\Delta(u)]$ as their generating function [7]

$$\begin{aligned} &\big[\,\bar{H}_m\,,\,\bar{H}_\ell\,\big] = 0 \quad , \quad \bar{H}_m \equiv \sum_{i=1}^{N}\big(\bar{D}'_i\big)^m \\ &\big[\,H_m\,,\,H_\ell\,\big] = 0 \quad , \quad H_m \equiv \Pi\big\{\bar{H}_m\big\} = \sum_{i=1}^{N}\Pi\Big\{\big(\bar{D}'_i\big)^m\Big\}\,. \end{aligned} \tag{2.41}$$

By virtue of (2.39), this new set is obviously equivalent to the $\{J_m\}$ set and from now on, we will focus on the sets $\{I_m\}$ and $\{H_m\}$.

3. The Dynamical Calogero Model

In order to obtain the Hamiltonian spin-chain conservation laws, one must first consider an N-body dynamical Calogero model in which the particles are chosen to have unit mass and are allowed to move along the line, under the influence of a position-space exchange interaction and subject to an harmonic confinement

$$\bar{\mathcal{H}}^{(\mathrm{CC})} \equiv \frac{1}{2}\sum_{i=1}^{N}P_i^2 - \frac{1}{2}\sum_{i,j=1}^{N}{}' \,\frac{g\big(g-\hbar\hat{K}_{ij}\big)}{R_{ij}R_{ji}} + \frac{1}{2}\omega^2\sum_{i=1}^{N}R_i^2\,. \tag{3.1}$$

The integrability of this model has been demonstrated e.g., in [20] by means of the operators

$$\bar{\mathcal{D}}_j^{\pm} \equiv P_j + \sqrt{-1}\,g\sum_{\substack{k=1\\k\neq j}}^{N}\frac{1}{R_{jk}}\hat{K}_{jk} \pm \sqrt{-1}\,\omega R_j\,, \tag{3.2}$$

which satisfy $\big(\bar{\mathcal{D}}_j^{\pm}\big)^{\dagger} = \bar{\mathcal{D}}_j^{\mp}$, in addition to the commutation properties

$$\begin{aligned} &\big[\,\bar{\mathcal{D}}_i^{\pm}\,,\,\bar{\mathcal{D}}_j^{\pm}\,\big] = 0 \\ &\big[\,\bar{\mathcal{D}}_i^{\pm}\,,\,\bar{\mathcal{D}}_j^{\mp}\,\big] = \mp 2\omega\delta_{ij}\Bigg(\hbar + g\sum_{\substack{k=1\\k\neq i}}^{N}\hat{K}_{ik}\Bigg) \pm (1-\delta_{ij})2\omega g\hat{K}_{ij}\,. \end{aligned} \tag{3.3}$$

From these, one can define [21] the Dunkl operator $\bar{\mathcal{D}}_i \equiv \bar{\mathcal{D}}_i^+ \bar{\mathcal{D}}_i^- + \hbar\omega$

$$
\begin{aligned}
\bar{\mathcal{D}}_i = -g^2 \sum_{\substack{j,k=1 \\ i\neq j\neq k\neq i}}^{N} \frac{1}{R_{jk}R_{ki}} \hat{K}_{ijk} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Big\{ \sqrt{-1}\, g \frac{1}{R_{ij}} \big(P_i + P_j\big) - \omega g \Big\} \hat{K}_{ij} \\
+ P_i^2 - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{g(g - \hbar \hat{K}_{ij})}{R_{ij}R_{ji}} + \omega^2 R_i^2
\end{aligned}
\tag{3.4}
$$

(where here and hereafter, we use the notation $\hat{K}_{i_1 \ldots i_n} \equiv \prod_{j=1}^{n-1} \hat{K}_{i_j i_{j+1}}$). The deformation parameter of this Dunkl operator is $\lambda = -2\omega g$; it therefore satisfies the commutation relation:

$$
\big[\, \bar{\mathcal{D}}_i \,,\, \bar{\mathcal{D}}_j \,\big] = -2\omega g \big(\bar{\mathcal{D}}_i - \bar{\mathcal{D}}_j\big) \hat{K}_{ij} \,. \tag{3.5}
$$

The representation of the yangian algebra induced by this Dunkl operator is given by

$$
\begin{aligned}
\mathcal{Q}_{\mathbf{0}} &= -\sum_{i=1}^{N} \mathbf{E}_i^{\top} \\
\mathcal{Q}_{\mathbf{1}} &= g^2 \sum_{i,j,k=1}^{N}{}' (\mathbf{E}_i \mathbf{E}_j \mathbf{E}_k)^{\top} \frac{1}{R_{ij}R_{jk}} + \sum_{i,j=1}^{N}{}' (\mathbf{E}_i \mathbf{E}_j)^{\top} \Big\{ \frac{\hbar g}{R_{ij}^2} - \frac{\sqrt{-1}\, g}{R_{ij}} \big(P_i + P_j\big) \Big\} \\
&\qquad - \sum_{i=1}^{N} \mathbf{E}_i^{\top} \Big\{ P_i^2 + g^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{R_{ij}^2} + \omega^2 R_i^2 \Big\} - N\omega g \mathbf{1} \,.
\end{aligned}
\tag{3.6}
$$

Its associated monodromy matrix then allows us to generate two non-trivial involutive sets of operators, denoted $\bar{\mathcal{I}}_m$ and $\bar{\mathcal{H}}_m$ (calligraphic symbols being used for the charges pertaining to the dynamical model). Their first member is given explicitly by

$$
\bar{\mathcal{I}}_1 = \sum_{i=1}^{N} \bar{\mathcal{D}}_i = 2\bar{\mathcal{H}}^{(\mathrm{CC})} - \omega g \sum_{i,j=1}^{N} \hat{K}_{ij} \tag{3.7}
$$

$$
\bar{\mathcal{H}}_1 = \sum_{i=1}^{N} \Big(\bar{\mathcal{D}}_i + 2\omega g \sum_{\substack{j=1 \\ j<i}}^{N} \hat{K}_{ij} \Big) = 2\bar{\mathcal{H}}^{(\mathrm{CC})} \,. \tag{3.8}
$$

Because $\bar{\mathcal{H}}^{(\mathrm{CC})}$ is included in $\{\bar{\mathcal{H}}_m\}$ and these essentially arise from an expansion of the quantum determinant, the symmetry of $\bar{\mathcal{H}}^{(\mathrm{CC})}$ under the monodromy matrix induced by this Dunkl operator is manifest. This means that both the $\bar{\mathcal{I}}_m$ and $\bar{\mathcal{H}}_m$ sets constitute

involutive invariants for this dynamical model. Moreover, the basic relations (3.3) can be shown [20] to imply

$$\left[\, \bar{\mathcal{H}}^{(\mathrm{CC})}\,,\, (\bar{\mathcal{D}}_k^{\pm})^n \,\right] = \pm n\hbar\omega (\bar{\mathcal{D}}_k^{\pm})^n\,, \tag{3.9}$$

thereby furnishing a set of creation operators from which the spectrum can be readily obtained.

4. The Frahm-Polychronakos spin chain

The Frahm-Polychronakos spin-chain model is defined by (1.2). In this expression, the x_i's are the zeroes of the Hermite polynomials. In this section, we will consider a generalized version of the FP Hamiltonian $H(\mathbf{r})$:

$$H(\mathbf{r}) = \frac{1}{2}\sum_{i,j=1}^{N}{}' \frac{1}{r_{ij}r_{ji}} P_{ij}\,. \tag{4.1}$$

in which $\mathbf{r}$ is a set of unconstrained position eigenvalues, in order to see explicitly how the yangian symmetry picks up the particular FP model, i.e. how it enforces $r_i = x_i$. In the following, the (potential) conserved charges pertaining to this general version of the spin chain that are inherited from the dynamical model will be denoted by $\mathcal{I}_m^{(0)}(\mathbf{r})$ and $\mathcal{H}_m^{(0)}(\mathbf{r})$. The subindex 0 refers to an $\hbar$-expansion to be explained shortly.

In order to generate a candidate symmetry algebra for this generalized model, we consider the position-space representation of the dynamical Calogero model, in which, from now on, we set $\omega = g = 1$. The spin part of $\bar{\mathcal{H}}^{(\mathrm{CC})}$ is simply isolated as the linear $\hbar$-piece of the Hamiltonian. More generally, the spin part is obtained by differentiating the Hamiltonian with respect to $\hbar$ and, since the kinetic term is quadratic in $\hbar$, setting $\hbar = 0$ at the end. This ignores the fact that the zeroes should be fixed at particular positions, but nevertheless suggests to consider the $\hbar$-expansion of the Dunkl operators and the related conserved operators.

Let us then expand the dynamical Dunkl operator (3.4) according to $\bar{\mathcal{D}}_i = \sum_k \bar{\mathcal{D}}_i^{(k)}\hbar^k$ and for the time being, concentrate on the zeroth order term:

$$\bar{\mathcal{D}}_i^{(0)}(\mathbf{r}) = -\sum_{\substack{j,k=1 \\ i\neq j\neq k\neq i}}^{N} \frac{1}{r_{jk}r_{ki}} K_{ijk} - \sum_{\substack{j=1 \\ j\neq i}}^{N} K_{ij} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{r_{ij}^2} + r_i^2\,. \tag{4.2}$$

Since the Dunkl algebra (3.5) is satisfied for all values of $\hbar$, $\bar{\mathcal{D}}_i^{(0)}(\mathbf{r})$ is also a genuine Dunkl operator, with deformation parameter $\lambda = -2$. Now for generic values of the r_i's, the induced $Y[su(n)]$ representation is irreducible and its quantum determinant is therefore a trivial c-number. As a corollary, the $\{\mathcal{H}_m^{(0)}(\mathbf{r})\}$ do not provide non-trivial conserved charges, i.e. these quantities are independent of any exchange operators. On the other hand, the set $\{\mathcal{I}_n^{(0)}(\mathbf{r})\}$ does provide a non-trivial involutive ensemble, its first member being given by

$$\mathcal{I}_1^{(0)}(\mathbf{r}) = \sum_{i=1}^{N} \Pi\Big\{\bar{\mathcal{D}}_i^{(0)}(\mathbf{r})\Big\} = -\sum_{i,j=1}^{N}{}' P_{ij} + \sum_{i,j=1}^{N}{}' \frac{1}{r_{ij}^2} + \sum_{i=1}^{N} r_i^2 \,. \tag{4.3}$$

To obtain this result, we used the identity

$$\sum_{j,k,\ell=1}^{N}{}' \frac{1}{r_{jk}r_{k\ell}} = \frac{1}{3}\sum_{j,k,\ell=1}^{N}{}' \left\{\frac{1}{r_{jk}r_{k\ell}} + \frac{1}{r_{k\ell}r_{\ell j}} + \frac{1}{r_{\ell j}r_{jk}}\right\} \equiv 0 \,. \tag{4.4}$$

Now, the higher order $\mathcal{I}_m^{(0)}(\mathbf{r})$ do not contain any term having the form of $H(\mathbf{r})$. In other words, this set of commuting operators has no relation at this point with the generalized model defined by the Hamiltonian $H(\mathbf{r})$. In order for the set $\{\mathcal{I}_m^{(0)}(\mathbf{r})\}$ to represent involutive invariants for (4.1), one must enforce the invariance of $H(\mathbf{r})$ under the corresponding $Y[su(n)]$ algebra, whose lower-order generators are given by

$$\begin{aligned}
\mathcal{Q}_{\mathbf{0}}^{(0)}(\mathbf{r}) &= -\sum_{i=1}^{N} \mathbf{E}_i^{\top} \\
\mathcal{Q}_{\mathbf{1}}^{(0)}(\mathbf{r}) &= \sum_{i,j,k=1}^{N}{}' (\mathbf{E}_i\mathbf{E}_j\mathbf{E}_k)^{\top} \frac{1}{r_{ij}r_{jk}} - \sum_{i=1}^{N} \mathbf{E}_i^{\top} \left\{ \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{r_{ij}^2} + r_i^2 \right\} - N\mathbf{1} \,.
\end{aligned} \tag{4.5}$$

In other words, we require that $\big[\, H(\mathbf{r})\, ,\, \mathcal{Q}_{(\mathbf{0},\mathbf{1})}^{(0)}(\mathbf{r}) \,\big] = 0$. A direct calculation [21] shows that this holds if and only if the variables r_i obey

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{r_{ij}^3} = \frac{1}{2} r_i \,. \tag{4.6}$$

One can show that this condition is satisfied by the zeroes (written x_i) of the Hermite polynomial $H_N(x)$ (cf. Appendix B). In fact, by judiciously substracting known summation identities [22] for these numbers, one can generate a whole sequence of 'higher order'

identities, the simplest of them being listed in Appendix B; these will play a crucial role in subsequent calculations.

In retrospect, by freezing the positions of the particles on the zeroes of $H_N(x)$, we send $H(\mathbf{r})$ on $H^{(\mathrm{FP})}$ and thus obtain an integrable $Y[su(n)]$-symmetric spin chain with a non-trivial involutive set of invariants given by

$$I_m(\mathbf{x}) \equiv \lim_{\mathbf{r}\to\mathbf{x}} \mathcal{I}_m^{(0)}(\mathbf{r}) = \sum_{i=1}^{N} \Pi\Big\{ \Big(\bar{\mathcal{D}}_i^{(0)}(\mathbf{x})\Big)^m \Big\} \,. \tag{4.7}$$

These are the conserved quantities first found by Polychronakos [13] (but without the yangian interpretation).

Moreover, defining $\mathcal{C}_m^{\pm} \equiv \sum_{i=1}^{N} \mathbf{E}_i \big(\bar{\mathcal{D}}_i^{\pm}\big)^m$, expanding (3.9) to $\mathcal{O}(\hbar)$ and setting $\omega = g = 1$, we find

$$\big[\, \bar{\mathcal{H}}^{(\mathrm{CC})(0)}(\mathbf{r}) \,,\, \mathcal{C}_m^{\pm(1)}(\mathbf{r}) \,\big] + \big[\, \bar{\mathcal{H}}^{(\mathrm{CC})(1)}(\mathbf{r}) \,,\, \mathcal{C}_m^{\pm(0)}(\mathbf{r}) \,\big] = \pm m\, \mathcal{C}_m^{\pm(0)}(\mathbf{r}) \,. \tag{4.8}$$

Because $\bar{\mathcal{H}}^{(\mathrm{CC})(0)}(\mathbf{r})$ is scalar and K_{ij}-invariant, the first commutator on the left hand side reduces to the action of the derivative on $\bar{\mathcal{H}}^{(\mathrm{CC})(0)}(\mathbf{r})$, which is given by

$$\frac{\partial}{\partial r_k} \Big\{ \bar{\mathcal{H}}^{(\mathrm{CC})(0)}(\mathbf{r}) \Big\} = \frac{1}{2} \frac{\partial}{\partial r_k} \Big\{ {\sum_{i,j=1}^{N}}' \frac{1}{r_{ij}^2} + \sum_{i=1}^{N} r_i^2 \Big\} = r_k - 2 \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{1}{r_{jk}^3} \,. \tag{4.9}$$

This vanishes as $\mathbf{r} \to \mathbf{x}$ (cf. the identity (B.7)) and (4.8) takes the form

$$\Big[\, \frac{1}{2} {\sum_{i,j=1}^{N}}' \frac{1}{x_{ij} x_{ji}} K_{ij} \,,\, \mathcal{C}_m^{\pm(0)}(\mathbf{x}) \,\Big] = \pm m\, \mathcal{C}_m^{\pm(0)}(\mathbf{x}) \,. \tag{4.10}$$

Taking now the projection and using the $K_{ij}P_{ij}$-invariance of the two commuted operators, we obtain a whole set of creation operators for the FP model:

$$\big[\, H^{(\mathrm{FP})} \,,\, \mathbf{C}_m^{\pm} \,\big] = \pm m\, \mathbf{C}_m^{\pm} \quad , \quad \mathbf{C}_m^{\pm} \equiv \Pi \Big\{ \sum_{k=1}^{N} \mathbf{E}_k \sum_{\substack{\ell=1\\ \ell\neq k}}^{N} \Big(\frac{1}{x_{k\ell}} K_{k\ell} \pm x_k \Big)^m \Big\} \,. \tag{4.11}$$

These generalize the lower-order creation operators found in [21,14]. We therefore possess a set of non-trivial creation operators $\mathbf{C}_m^{\pm}$ and conservation laws I_m. However, as previously pointed out, the $\{\mathcal{H}_m^{(0)}\}$ set associated with the symmetry algebra is trivial. Since $\{\mathcal{I}_m^{(0)}\}$ does not contain the defining Hamiltonian, a whole set of commuting conservation laws is still missing.

5. The Hamiltonian conservation laws of the FP model

Our proof for the commutativity of the conservation laws will strongly rely on the structure of the FP Hilbert space. For the $su(2)$ Haldane-Shastry model, the yangian symmetry algebra has been shown to be a direct sum of irreducible $Y[su(2)]$ "motif" representations, each possible motif appearing with unit multiplicity [18] . This result has been obtained by calculating the dimensions of the $Y[su(2)]$ motif representations as a tensor product of $su(2)$ spin representations and then showing that these motifs exhaust the Hilbert space. For the $su(n)$ case ($n > 2$), the motifs are not expressible as a free tensor product [21] and to our knowledge, it hasn't been proved that the $Y[su(n)]$ motifs exhaust the Hilbert space. However, strong numerical evidence [21] suggests that the symmetry algebras for both the $su(n)$ HS and FP models are also a direct sum of $non-degenerate$ motifs. In the following, we will consider this statement to be true.

The non-degenerate character of the motifs implies that any two operators A and B commuting with the monodromy matrix $\mathbf{T}(u)$ must also commute amongst themselves (see e.g., [7]). Indeed, the Hilbert space of our reducible $Y[su(n)]$ invariant theory contains a certain number of yangian highest-weight states, each of which is associated with a given motif. These highest-weight states are eigenvectors of the diagonal elements $T^{aa}(u)$ ($a = 1 \dots n$), with eigenvalues that completely specify the given motif. Since the motifs have unit multiplicity, the highest-weight states $T^{aa}(u)$-eigenvalues form non-degenerate sets. Now consider the two states $AB \mid \Lambda >$ and $BA \mid \Lambda >$, where $\mid \Lambda >$ is a yangian highest-weight state. Since A and B commute with $\mathbf{T}(u)$, both these states will be eigenvectors of $T^{aa}(u)$ with the same eigenvalue. But since these eigenvalues are non-degenerate, the two states must in fact be proportional to one another, which implies $\left[A \, , \, B \right] = 0$ on any highest-weight state. In fact, since all of the states can be generated by acting on the highest-weight states with lowering operators of the form $\prod_i T^{a_i b_i}(\lambda_i)$ (with $a_i < b_i$ and the λ_i chosen to satisfy a set of Bethe ansatz equations), one sees that A and B will in fact commute in the entire Hilbert space.

We will now prove that the first order terms in the $\hbar$-expansion of the dynamical $\{\mathcal{H}_m\}$ set satisfy the FP $Y[su(n)]$ symmetry and are therefore in involution. Starting from the dynamical symmetry $\left[\mathcal{Q}_{(\mathbf{0},\mathbf{1})}(\mathbf{r}) \, , \, \mathcal{H}_m(\mathbf{r}) \right] = 0$ and expanding to $\mathcal{O}(\hbar)$, we have

$$\left[\mathcal{Q}^{(0)}_{(\mathbf{0},\mathbf{1})}(\mathbf{r}) \, , \, \mathcal{H}^{(1)}_m(\mathbf{r}) \right] = -\left[\mathcal{Q}^{(1)}_{(\mathbf{0},\mathbf{1})}(\mathbf{r}) \, , \, \mathcal{H}^{(0)}_m(\mathbf{r}) \right] . \tag{5.1}$$

On the right hand side, the commutation with $\mathcal{Q}^{(1)}_{\mathbf{0}}$ is trivially zero, while that with $\mathcal{Q}^{(1)}_{\mathbf{1}}$ can be greatly simplified by appealing to the scalar nature of $\mathcal{H}^{(0)}_m(\mathbf{r})$ and its invariance under K_{ij} and $K_{ij}P_{ij}$:

$$\left[\,\mathcal{Q}^{(0)}_{\mathbf{1}}(\mathbf{r})\,,\,\mathcal{H}^{(1)}_m(\mathbf{r})\,\right] = {\sum_{i,j=1}^{N}}'(\mathbf{E}_i\mathbf{E}_j)^{\top}\frac{1}{r_{ij}}\left\{\frac{\partial}{\partial r_i}\Big(\mathcal{H}^{(0)}_m(\mathbf{r})\Big)+\frac{\partial}{\partial r_j}\Big(\mathcal{H}^{(0)}_m(\mathbf{r})\Big)\right\}. \tag{5.2}$$

To further simplify the right hand side, we will now explicitly calculate $\mathcal{H}^{(0)}_m(\mathbf{r})$. To this end, let us return to the abstract Hilbert-space formalism and consider the following integral

$$F_m(\mathbf{r}) \equiv \int dy_1\dots dy_N\ \ < y^{(1)}_1\!\dots y^{(N)}_N \mid \mathcal{H}^{(0)}_m(R_1\dots R_N)\mid \mathrm{sym}(\mathbf{r})>\ , \tag{5.3}$$

where $\mid \mathrm{sym}(\mathbf{r})> \equiv \sum_{\sigma\in S_N}\ \mid r^{(1)}_{\sigma(1)}\dots r^{(N)}_{\sigma(N)}>$ and the notation $\mathcal{H}^{(0)}_m(R_1\dots R_N)$ is used to stress that these charges are K_{ij}-independent. Applying $\mathcal{H}^{(0)}_m(R_1\dots R_N)$ to the left yields

$$\begin{aligned}F_m(\mathbf{r}) &= \sum_{\sigma\in S_N}\int dy_1\dots dy_N\,\mathcal{H}^{(0)}_m(y_1\dots y_N)< y^{(1)}_1\!\dots y^{(N)}_N\mid r^{(1)}_{\sigma(1)}\dots r^{(N)}_{\sigma(N)}> \\ &= \sum_{\sigma\in S_N}\int dy_1\dots dy_N\,\mathcal{H}^{(0)}_m(y_1\dots y_N)\delta(y_1-r_{\sigma(1)})\dots\delta(y_N-r_{\sigma(N)}) \\ &= \sum_{\sigma\in S_N}\mathcal{H}^{(0)}_m(r_{\sigma(1)}\dots r_{\sigma(N)}) = N!\,\mathcal{H}^{(0)}_m(r_1\dots r_N)\,,\end{aligned} \tag{5.4}$$

where we have used the K_{ij}-invariance of $\mathcal{H}^{(0)}_m(r_1\dots r_N)$ in the last step. We therefore have the following result

$$\mathcal{H}^{(0)}_m(\mathbf{r}) = \frac{1}{N!}F_m(\mathbf{r})\,. \tag{5.5}$$

On the other hand, going back to (5.3), one can express $F_m(\mathbf{r})$ in the form

$$F_m(\mathbf{r}) = \int dy_1\dots dy_N\ \ < y^{(1)}_1\!\dots y^{(N)}_N \mid\ \sum_{i=1}^{N}\Big\{\bar{\mathcal{D}}'^{(0)}_i\Big\}^m \mid \mathrm{sym}(\mathbf{r})>\ , \tag{5.6}$$

where the modified Dunkl operator has the following explicit expression

$$\bar{\mathcal{D}}'^{(0)}_i = \bar{\mathcal{D}}^{(0)}_i + 2\sum_{\substack{j=1\\ j<i}}^{N}\hat{K}_{ij} = \sum_{\substack{j,k=1\\ i\neq j\neq k\neq i}}^{N}\frac{-1}{R_{jk}R_{ki}}\hat{K}_{ijk} + \sum_{\substack{j=1\\ j\neq i}}^{N}\mathrm{sgn}(i-j)\hat{K}_{ij} + \sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{R^2_{ij}} + R^2_i\,. \tag{5.7}$$

Applying now $\mathcal{H}^{(0)}_m(R_1\dots R_N)$ to the right and using

$$\hat{K}_{ij}\ \mid \mathrm{sym}(\mathbf{r})> =\ \mid \mathrm{sym}(\mathbf{r})> \tag{5.8}$$

$$f(R_i)\ \mid \mathrm{sym}(\mathbf{r})> = \sum_{\sigma\in S_N} f(r_{\sigma(i)})\ \mid r^{(1)}_{\sigma(1)}\dots r^{(N)}_{\sigma(N)}>\ , \tag{5.9}$$

one obtains

$$F_m(\mathbf{r}) = \sum_{\sigma\in S_N}\sum_{i=1}^{N}\Bigg\{\sum_{\substack{j,k=1\\ i\neq j\neq k\neq i}}^{N}\frac{-1}{r_{\sigma(j)\sigma(k)}r_{\sigma(k)\sigma(i)}} + \sum_{\substack{j=1\\ j\neq i}}^{N}\mathrm{sgn}(i-j) + \sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{r^2_{\sigma(i)\sigma(j)}} + r^2_{\sigma(i)}\Bigg\}^m . \tag{5.10}$$

Considering now the $(N-1)!$ permutations for which $\sigma(i)=\ell$, this can be rewritten as

$$F_m(\mathbf{r}) = (N-1)!\sum_{\ell=1}^{N}\sum_{i=1}^{N}\Bigg\{\sum_{\substack{j,k=1\\ \ell\neq j\neq k\neq \ell}}^{N}\frac{-1}{r_{jk}r_{k\ell}} + \sum_{\substack{j=1\\ j\neq i}}^{N}\mathrm{sgn}(i-j) + \sum_{\substack{j=1\\ j\neq \ell}}^{N}\frac{1}{r^2_{j\ell}} + r^2_{\ell}\Bigg\}^m . \tag{5.11}$$

Finally, using a binomial expansion to factorize the $\mathrm{sgn}(i-j)$ term (i.e., the r_i-independent piece) and using (5.5), we find

$$\mathcal{H}^{(0)}_m(\mathbf{r}) = \sum_{\ell=1}^{N}\sum_{p=0}^{m}C_{pm}\Bigg\{\sum_{\substack{j,k=1\\ \ell\neq j\neq k\neq \ell}}^{N}\frac{-1}{r_{jk}r_{k\ell}} + \sum_{\substack{j=1\\ j\neq \ell}}^{N}\frac{1}{r^2_{j\ell}} + r^2_{\ell}\Bigg\}^p , \tag{5.12}$$

where

$$C_{pm} \equiv \frac{1}{N}\binom{m}{p}\sum_{i=1}^{N}\Big\{2i-(N+1)\Big\}^{m-p} . \tag{5.13}$$

To complete the calculation of the commutator (5.1), we need to evaluate the action of the derivative on $\mathcal{H}^{(0)}_m(\mathbf{r})$ (cf. (5.2)) and freeze the particle positions:

$$\lim_{\mathbf{r}\to\mathbf{x}}\frac{\partial}{\partial r_i}\Big\{\mathcal{H}^{(0)}_m(\mathbf{r})\Big\} = \lim_{\mathbf{r}\to\mathbf{x}}\sum_{\ell=1}^{N}\sum_{p=0}^{m}C_{pm}\,p\Bigg\{\sum_{\substack{j,k=1\\ \ell\neq j\neq k\neq \ell}}^{N}\frac{-1}{x_{jk}x_{k\ell}} + \sum_{\substack{j=1\\ j\neq \ell}}^{N}\frac{1}{x^2_{j\ell}} + x^2_{\ell}\Bigg\}^{p-1} \cdot\frac{\partial}{\partial r_i}\Bigg\{\sum_{\substack{j,k=1\\ \ell\neq j\neq k\neq \ell}}^{N}\frac{-1}{r_{jk}r_{k\ell}} + \sum_{\substack{j=1\\ j\neq \ell}}^{N}\frac{1}{r^2_{j\ell}} + r^2_{\ell}\Bigg\} . \tag{5.14}$$

In such calculations, the $\mathbf{r}\to\mathbf{x}$ limit is not a simple substitution and must be taken with care. Indeed, the summation formulae for x_i are valid for *numbers* and are therefore not preserved by the action of the derivatives. This means that one may take the substitution $\mathbf{r}\to\mathbf{x}$ and use the simplifying identities only if the targeted expression is no longer acted upon by any derivatives. In light of this remark, we see that we cannot simplify the second

bracketed factor in (5.14) without first carrying out the differentiation. However, we can use the formulae (B.7), (B.8) and (B.11) to reduce the first bracketed factor right away:

$$\begin{aligned}\sum_{\substack{k=1\\k\neq\ell}}^{N}\frac{1}{x_{\ell k}}\Big(-\sum_{\substack{j=1\\j\neq k}}^{N}\frac{1}{x_{kj}}+\frac{1}{x_{k\ell}}\Big)+\sum_{\substack{j=1\\j\neq\ell}}^{N}\frac{1}{x_{j\ell}^2}+x_\ell^2 &= -\sum_{\substack{k=1\\k\neq\ell}}^{N}\frac{x_k}{x_{\ell k}}+x_\ell^2\\ &= -x_\ell^2+(N-1)+x_\ell^2=N-1\,.\end{aligned}\tag{5.15}$$

We are thus left with

$$\begin{aligned}&\lim_{\mathbf{r}\to\mathbf{x}}\frac{\partial}{\partial r_i}\Big\{\mathcal{H}_m^{(0)}(\mathbf{r})\Big\}=\\ &\quad\lim_{\mathbf{r}\to\mathbf{x}}\sum_{\ell=1}^{N}\sum_{p=0}^{m}C_{pm}\,p(N-1)^{p-1}\frac{\partial}{\partial r_i}\Big\{\sum_{\substack{j,k=1\\ \ell\neq j\neq k\neq\ell}}^{N}\frac{-1}{r_{jk}r_{k\ell}}+\sum_{\substack{j=1\\j\neq\ell}}^{N}\frac{1}{r_{j\ell}^2}+r_\ell^2\Big\}\,.\end{aligned}\tag{5.16}$$

Commuting the sum over ℓ past the derivative and using the identity (4.4) we finally obtain

$$\begin{aligned}\lim_{\mathbf{r}\to\mathbf{x}}\frac{\partial}{\partial r_i}\Big\{\mathcal{H}_m^{(0)}(\mathbf{r})\Big\}&=\lim_{\mathbf{r}\to\mathbf{x}}\sum_{p=0}^{m}C_{pm}\,p(N-1)^{p-1}\frac{\partial}{\partial r_i}\Big\{\sum_{\substack{\ell=1\\ \ell\neq i}}^{N}\frac{1}{r_{\ell i}^2}+\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{r_{ij}^2}+r_i^2\Big\}\\ &=\sum_{p=0}^{m}C_{pm}\,p(N-1)^{p-1}2\Big\{x_i-2\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{x_{ij}^3}\Big\}=0\,,\end{aligned}\tag{5.17}$$

where we have used relation (B.9) in the last step. Substituting this result back in (5.2), we see that the quantities $H_{2m}\equiv\lim_{\mathbf{r}\to\mathbf{x}}\mathcal{H}_m^{(1)}(\mathbf{r})$ satisfy the FP $Y[su(n)]$ symmetry. In other words, the H_{2m} all commute with the monodromy matrix. As already pointed out, this implies that they are necessarily in involution.

A compact but implicit expression for the H_{2m} following from the $\hbar$-expansion of the dynamical operators, is given by

$$H_{2m}=\lim_{\mathbf{r}\to\mathbf{x}}\sum_{i=1}^{N}\sum_{p=0}^{m-1}\Pi\Big\{A_i^p(\mathbf{r})B_i(\mathbf{r})A_i^{m-p-1}(\mathbf{r})\Big\}\,,\tag{5.18}$$

where

$$\begin{aligned}A_i(\mathbf{r})&\equiv\sum_{\substack{j,k=1\\ i\neq j\neq k\neq i}}^{N}\frac{-1}{r_{jk}r_{ki}}K_{ijk}+\sum_{\substack{j=1\\j\neq i}}^{N}\mathrm{sgn}(i-j)K_{ij}+\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{r_{ij}^2}+r_i^2\\ B_i(\mathbf{r})&\equiv\sum_{\substack{j=1\\j\neq i}}^{N}\Big\{\frac{1}{r_{ij}}\Big(\frac{\partial}{\partial r_i}+\frac{\partial}{\partial r_j}\Big)-\frac{1}{r_{ij}^2}\Big\}K_{ij}\,.\end{aligned}\tag{5.19}$$

The first two members of this set can be calculated as

$$H_2 = \sum_{i,j=1}^{N}{}' \frac{1}{x_{ij}x_{ji}} P_i j \tag{5.20}$$

$$\begin{aligned} H_4 = &\sum_{i,j,k\ell=1}^{N}{}' \left\{ \frac{-1}{x_{ij}x_{jk}x_{k\ell}x_{\ell i}} \right\} P_{ijk\ell} \\ &+ \sum_{i,j=1}^{N}{}' \left\{ \frac{2}{x_{ij}^4} - \frac{2}{3}\frac{x_i x_j}{x_{ij}^2} - \frac{8}{3}(N-1)\frac{1}{x_{ij}^2} - \frac{4}{3} \right\} P_{ij} \,. \end{aligned} \tag{5.21}$$

Because $H_2 = 2H^{(\mathrm{FP})}$, these involutive quantities are necessarily invariants of the FP model. Note that in the calculation of H_4, we have used the summation identities (B.8), (B.15) and (B.16) in order to simplify the final result. We have also been able to crosscheck the commutativity of H_2 and H_4 by a direct computation, the details of which are given in Appendix A. It should also be pointed out that since it is impossible in quantum mechanics to freeze the particle positions onto the lattice sites and enforce at the same time the vanishing of their momenta, the absence of derivatives in H_2 and H_4 should not be regarded as a consequence of the freezing procedure but as a rather impressive mathematical cancelation, whose *raison d'être* has yet to be determined. Also, notice that our limiting procedure does not generate any odd-type conservation laws because the dynamical Calogero model simply does not possess such symmetries. On the other hand, it is easy to see that the FP model does possess such symmetries by verifying explicitly that it commutes with the following operators

$$H_1 = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \tag{5.22}$$

$$H_3 = \sum_{i,j,k=1}^{N}{}' \Big(\frac{1}{x_{ij}x_{jk}x_{ki}} \Big) P_{ijk} - \frac{3}{2} \sum_{i,j=1}^{N}{}' \Big(\frac{1}{x_{ij}x_{ji}} \Big) \frac{\partial}{\partial x_i} \,. \tag{5.23}$$

Here, we have not performed any simplifications on H_3, in order to show that $\big[\, H_1 \,,\, H_3 \,\big] = 0$ (recall that the summation identities can only be used once all derivatives have been commuted to the right). It therefore seems possible to generate odd-type H_m, albeit by brute force. These odd conservation laws seem to commute amongst themselves as well as with the even H_m although they manifestly do not possess the yangian symmetry. This means that Haldane and Talstra's argument cannot be used to isolate their generating function; we have not yet found the generating function for such a set.

6. Conclusion

Using an $\hbar$-expansion of the dynamical Calogero model, we have succeeded in constructing an even set $\{H_2, H_4 \ldots\}$ of involutive charges for the Frahm-Polychronakos spin chain, following to a large extent the procedure of [7]. However, as these authors pointed out, we should stress that the underlying $\hbar$-expansion constitutes a somewhat ad-hoc procedure and does not seem to shed much light on the fundamental origin of these conservation laws. One wonders whether the complicated limiting procedure is really necessary and whether these invariants could not be generated in a simpler way, from an intrinsic spin-chain formulation. In addition, we could ask whether explicit expressions for these Hamiltonian conservation laws could be written, in analogy with those of the XXX model [23]. We definitely see a similar pattern emerging but the expressions for the relative coefficients of the various terms appear rather complicated. Finally, a brute force computation of H_1 and H_3 seems to hint at the existence of an odd set of involutive charges which does not obey the yangian symmetry, and for which we still lack a generating function.

Appendix A. A direct computation of $\left[\, H_2\, , \, H_4 \,\right]$

In this appendix, we show that the commutator $\left[\, H_2\, , \, H_4 \,\right]$ vanishes by calculating it explicitly. For compactness, let us start by expressing the conservation laws in the form

$$H_2 = {\sum_{i,j=1}^{N}}' h_{ij} P_{ij} \tag{A.1}$$

$$H_4 = -{\sum_{i,j,k,\ell=1}^{N}}' h_{ijk\ell} P_{ijk\ell} + {\sum_{i,j=1}^{N}}' f_{ij} P_{ij}\ , \tag{A.2}$$

where

$$h_{ij} = \frac{1}{x_{ij} x_{ji}} \tag{A.3}$$

$$h_{ijk\ell} = \frac{1}{x_{ij} x_{jk} x_{k\ell} x_{\ell i}} \tag{A.4}$$

$$f_{ij} = \frac{2}{x_{ij}^4} - \frac{2}{3}\frac{x_i x_j}{x_{ij}^2} - \frac{8}{3}(N-1)\frac{1}{x_{ij}^2} - \frac{4}{3}\,. \tag{A.5}$$

A direct calculation yields the following commutator

$$\begin{aligned}\big[\, H_2 \,,\, H_4 \,\big] = 8 \sum_{i,j,k,\ell,m}^{N}{}' (h_{ij} - h_{im}) h_{jk\ell m} P_{ijk\ell m} \\ - 4 \sum_{i,j,k,\ell}^{N}{}' (h_{ik} - h_{j\ell}) h_{ijk\ell} P_{ij} P_{k\ell} \\ - 4 \sum_{i,j,k}^{N}{}' \bigg\{ 2 \sum_{\substack{\ell=1 \\ \ell \neq i,j,k}}^{N} (h_{i\ell} - h_{k\ell}) h_{ijk\ell} - (h_{ik} - h_{jk}) f_{ij} \bigg\} P_{ijk} \,.\end{aligned} \tag{A.6}$$

Defining now the cyclic sum operator

$$\begin{aligned}\sum_{\{i_1 \ldots i_k\}}^{\text{cyclic}} f(x_{i_1} \ldots x_{i_k}) = f(x_{i_1} \ldots x_{i_k}) + f(x_{i_2}, x_{i_3} \ldots x_{i_k}, x_{i_1}) \\ + \ldots + f(x_{i_k}, x_{i_1} \ldots x_{i_{k-1}})\end{aligned} \tag{A.7}$$

and using the fact that the exchange operators in (A.6) are invariant under cyclic permutations of their indices, we can write

$$\begin{aligned}\big[\, H_2 \,,\, H_4 \,\big] = 8 \sum_{i,j,k,\ell,m}^{N}{}' \frac{1}{5} \sum_{\{i,j,k,\ell,m\}}^{\text{cyclic}} \Big\{ (h_{ij} - h_{im}) h_{jk\ell m} \Big\} P_{ijk\ell m} \\ - 4 \sum_{i,j,k,\ell}^{N}{}' \frac{1}{4} \sum_{\{i,j\}}^{\text{cyclic}} \sum_{\{k,\ell\}}^{\text{cyclic}} \Big\{ (h_{ik} - h_{j\ell}) h_{ijk\ell} \Big\} P_{ij} P_{k\ell} \\ - 4 \sum_{i,j,k}^{N}{}' \frac{1}{3} \sum_{\{i,j,k\}}^{\text{cyclic}} \bigg\{ 2 \sum_{\substack{\ell=1 \\ \ell \neq i,j,k}}^{N} (h_{i\ell} - h_{k\ell}) h_{ijk\ell} - (h_{ik} - h_{jk}) f_{ij} \bigg\} P_{ijk} \,.\end{aligned} \tag{A.8}$$

This commutator will therefore vanish if one can prove that

$$F_1 \equiv \sum_{\{i,j,k,\ell,m\}}^{\text{cyclic}} \Big\{ (h_{ij} - h_{im}) h_{jk\ell m} \Big\} = 0 \tag{A.9}$$

$$F_2 \equiv \sum_{\{i,j\}}^{\text{cyclic}} \sum_{\{k,\ell\}}^{\text{cyclic}} \Big\{ (h_{ik} - h_{j\ell}) h_{ijk\ell} \Big\} = 0 \tag{A.10}$$

$$F_3 \equiv \sum_{\{i,j,k\}}^{\text{cyclic}} \bigg\{ -2 \sum_{\substack{\ell=1 \\ \ell \neq i,j,k}}^{N} (h_{i\ell} - h_{k\ell}) h_{ijk\ell} + (h_{ik} - h_{jk}) f_{ij} \bigg\} = 0 \,. \tag{A.11}$$

The first condition (A.9) is shown to be satisfied in the following manner. First, we extract a cyclic invariant from the sum

$$\begin{aligned} F_1 &= \sum_{\{i,j,k,\ell,m\}}^{\text{cyclic}} \left\{ \left(\frac{1}{x_{ij}^2} - \frac{1}{x_{im}^2} \right) \frac{1}{x_{jk} x_{k\ell} x_{\ell m} x_{mj}} \right\} \\ &= \frac{1}{x_{ij} x_{jk} x_{k\ell} x_{\ell m} x_{mi}} \sum_{\{i,j,k,\ell,m\}}^{\text{cyclic}} \left\{ \frac{x_{mi}}{x_{ij} x_{mj}} - \frac{x_{ij}}{x_{mi} x_{mj}} \right\} \\ &\equiv h_{ijk\ell m} \sum_{\{i,j,k,\ell,m\}}^{\text{cyclic}} \left\{ \frac{x_{mi}}{x_{ij} x_{mj}} - \frac{x_{ij}}{x_{mi} x_{mj}} \right\}. \end{aligned} \tag{A.12}$$

The next few steps are just basic algebra

$$\begin{aligned} F_1 &= h_{ijk\ell m} \sum_{\{i,j,k,\ell,m\}}^{\text{cyclic}} \left\{ \frac{x_{mi}^2 - x_{ij}^2}{x_{ij} x_{mj} x_{mi}} \right\} \\ &= h_{ijk\ell m} \sum_{\{i,j,k,\ell,m\}}^{\text{cyclic}} \left\{ \frac{(x_m + x_j) x_{mj} - 2 x_i x_{mj}}{x_{ij} x_{mj} x_{mi}} \right\} \\ &= h_{ijk\ell m} \sum_{\{i,j,k,\ell,m\}}^{\text{cyclic}} \left\{ \frac{x_m + x_j - 2x_i}{x_{ij} x_{mi}} \right\}. \end{aligned} \tag{A.13}$$

Extracting once more the cyclic invariant $h_{ijk\ell m}$ gives

$$F_1 = h_{ijk\ell m}^2 \sum_{\{i,j,k,\ell,m\}}^{\text{cyclic}} \left\{ (x_m + x_j - 2x_i) x_{jk} x_{k\ell} x_{\ell m} \right\}. \tag{A.14}$$

This cyclic sum can then be shown to vanish by plainly writing down all of its terms. The second condition (A.10) can be proved to hold in a similar fashion. Establishing the vanishing of F_3 is a bit more tricky however. The main steps are as follows. First, we write F_3 explicitly:

$$\begin{aligned} F_3 = \sum_{\{i,j,k\}}^{\text{cyclic}} \Bigg\{ & \frac{2}{x_{ij} x_{jk}} \Bigg[\sum_{\substack{\ell=1 \\ \ell \neq i,k}} \left(\frac{1}{x_{k\ell} x_{\ell i}^3} - \frac{1}{x_{i\ell} x_{\ell k}^3} \right) - \frac{1}{x_{kj} x_{ji}^3} + \frac{1}{x_{ij} x_{jk}^3} \Bigg] \\ & + \left(\frac{1}{x_{jk}^2} - \frac{1}{x_{ik}^2} \right) \left(\frac{2}{x_{ij}^4} - \frac{2}{3} \frac{x_i x_j}{x_{ij}^2} - \frac{8}{3} (N-1) \frac{1}{x_{ij}^2} - \frac{4}{3} \right) \Bigg\}. \end{aligned} \tag{A.15}$$

We start by using the summation identity (B.17) in order to simplify the first two terms and notice that the last two terms in the last parenthesis do not contribute. The reduced

expression is:

$$F_3 = \sum_{\{i,j,k\}}^{\text{cyclic}} \left\{ \frac{2}{x_{ij}x_{jk}} \left[\frac{1}{3} \frac{x_k^2 - x_i^2}{x_{ki}^2} - \frac{1}{2} \frac{x_i + x_k}{x_{ki}} - \frac{1}{x_{kj}x_{ji}^3} + \frac{1}{x_{ij}x_{jk}^3} \right] \right.$$
$$\left. + \frac{2}{x_{ij}^4 x_{jk}^2} - \frac{2}{x_{ij}^4 x_{ik}^2} + \frac{2}{3} \frac{x_i x_j}{x_{ij}^2 x_{ik}^2} - \frac{2}{3} \frac{x_i x_j}{x_{ij}^2 x_{jk}^2} \right\}. \tag{A.16}$$

The cyclic sum will also cancel the last two terms in square brackets with the subsequent two terms so that we are left with

$$\begin{aligned} F_3 &= \sum_{\{i,j,k\}}^{\text{cyclic}} \left\{ -\frac{1}{3} \frac{x_i + x_k}{x_{ij}x_{jk}x_{ki}} + \frac{2}{3} \frac{x_i x_j}{x_{ij}^2 x_{ik}^2} - \frac{2}{3} \frac{x_i x_j}{x_{ij}^2 x_{jk}^2} \right\} \\ &= -\frac{1}{3} \frac{1}{x_{ij}x_{jk}x_{ki}} \sum_{\{i,j,k\}}^{\text{cyclic}} (x_i + x_k) + \frac{2}{3} \frac{1}{x_{ij}^2 x_{jk}^2 x_{ki}^2} \sum_{\{i,j,k\}}^{\text{cyclic}} x_i x_j (x_{jk}^2 - x_{ki}^2) \\ &= -\frac{2}{3} \frac{(x_i + x_j + x_k)}{x_{ij}x_{jk}x_{ki}} + \frac{2}{3} \frac{(x_i x_j^3 - x_i^3 x_j + x_j x_k^3 - x_j^3 x_k + x_k x_i^3 - x_i x_k^3)}{x_{ij}^2 x_{jk}^2 x_{ki}^2} \\ &= -\frac{2}{3} \frac{(x_i + x_j + x_k)}{x_{ij}x_{jk}x_{ki}} + \frac{2}{3} \frac{(x_i + x_j + x_k) x_{ij}x_{jk}x_{ki}}{x_{ij}^2 x_{jk}^2 x_{ki}^2} = 0 \, . \end{aligned} \tag{A.17}$$

We therefore see that H_2 and H_4 do commute, a fact which corroborates the validity of the dynamical $\hbar$-expansion used throughout this work.

Appendix B. The zeroes of the Hermite polynomials: summation identities

In this appendix, we briefly show how the lattice sites of the FP model, defined by (4.6), can be identified with the zeroes of the Hermite polynomial $H_N(x)$ and then present a series of summation identities which are vital for the reduction of certain expressions. Following [14], consider then the Hermite differential equation

$$H_N''(x) - 2xH_N'(x) + 2NH_N(x) = 0 \, . \tag{B.1}$$

Letting $x_i (i = 1 \ldots N)$ denote the zeroes of $H_N(x)$ and evaluating (B.1) at an arbitrary zero x_ℓ gives

$$H_N''(x_\ell) = 2x_\ell H_N'(x_\ell) \, . \tag{B.2}$$

Factorizing $H_N(x)$ in terms of its zeroes and substituting in (B.2) generates the identity

$$\sum_{\substack{k=1\\k\neq j}}^{N} \frac{1}{x_{jk}} = x_j\,. \tag{B.3}$$

As already mentioned, a number of simple summation identities generalizing the previous one have already been discovered some time ago [22]. One can easily generate more complicated formulae. The general procedure is the following: to increment a power to the numerator of (B.3), we can proceed as follows

$$\begin{aligned}
\sum_{\substack{k=1\\k\neq j}}^{N} \frac{x_{jk}}{x_{jk}} = (N-1) \quad &\Longrightarrow \quad x_j \sum_{\substack{k=1\\k\neq j}}^{N} \frac{1}{x_{jk}} - \sum_{\substack{k=1\\k\neq j}}^{N} \frac{x_k}{x_{jk}} = (N-1) \\
&\Longrightarrow \quad \sum_{\substack{k=1\\k\neq j}}^{N} \frac{x_k}{x_{jk}} = x_j^2 - (N-1)\,.
\end{aligned} \tag{B.4}$$

On the other hand, to increase a power in the denominator, the procedure is

$$\begin{aligned}
&\sum_{\substack{k=1\\k\neq j}}^{N} \frac{1}{x_{jk}} - \sum_{\substack{k=1\\k\neq i}}^{N} \frac{1}{x_{ik}} = \sum_{\substack{k=1\\k\neq i,j}}^{N} \frac{x_{ik}-x_{jk}}{x_{jk}x_{ik}} + \frac{1}{x_{ji}} - \frac{1}{x_{ij}} = \sum_{\substack{k=1\\k\neq i,j}}^{N} \frac{x_{ij}}{x_{jk}x_{ki}} - \frac{2}{x_{ij}} \Longrightarrow \\
&\sum_{\substack{k=1\\k\neq i,j}}^{N} \frac{1}{x_{jk}x_{ik}} = -\frac{1}{x_{ij}} \left[\sum_{\substack{k=1\\k\neq j}}^{N} \frac{1}{x_{jk}} - \sum_{\substack{k=1\\k\neq i}}^{N} \frac{1}{x_{ik}} + \frac{2}{x_{ij}} \right] = -\frac{1}{x_{ij}} \left[x_j - x_i + \frac{2}{x_{ij}} \right],
\end{aligned} \tag{B.5}$$

that is

$$\sum_{\substack{k=1\\k\neq i,j}}^{N} \frac{1}{x_{jk}x_{ik}} = 1 - \frac{2}{x_{ij}^2}\,. \tag{B.6}$$

Using such procedures, one can generate a whole set of summation identities, the most useful of them being given by

$$\sum_{\substack{k=1\\k\neq i}}^{N} \frac{1}{x_{ik}} = x_i \tag{B.7}$$

$$\sum_{\substack{k=1\\k\neq i}}^{N} \frac{1}{x_{ik}^2} = \frac{2}{3}(N-1) - \frac{1}{3}x_i^2 \tag{B.8}$$

$$\sum_{\substack{k=1\\k\neq i}}^{N} \frac{1}{x_{ik}^3} = \frac{1}{2} x_i \tag{B.9}$$

$$\sum_{\substack{k=1\\k\neq i}}^{N} \frac{1}{x_{ik}^4} = \frac{1}{45} \Big[2(N+2) - x_i^2\Big] \Big[2(N-1) - x_i^2\Big] \tag{B.10}$$

$$\sum_{\substack{k=1\\k\neq i}}^{N} \frac{x_k}{x_{ik}} = x_i^2 - (N-1) \tag{B.11}$$

$$\sum_{\substack{k=1\\k\neq i}}^{N} \frac{x_k}{x_{ik}^2} = -\frac{1}{3} x_i^3 + \left[\frac{2}{3}(N-1) - 1\right] \tag{B.12}$$

$$\sum_{\substack{k=1\\k\neq i}}^{N} \frac{x_k}{x_{ik}^3} = \frac{5}{6} x_i^2 - \frac{2}{3}(N-1) \tag{B.13}$$

$$\sum_{\substack{k=1\\k\neq i}}^{N} \frac{x_k^2}{x_{ik}} = x_i^3 - (N-2) x_i \tag{B.14}$$

$$\sum_{\substack{k=1\\i\neq j\neq k\neq i}}^{N} \frac{1}{x_{ik} x_{kj}} = 1 - \frac{2}{x_{ij}^2} \tag{B.15}$$

$$\sum_{\substack{k=1\\i\neq j\neq k\neq i}}^{N} \frac{1}{x_{ik} x_{kj}^2} = \frac{1}{3} \frac{2N+1-x_j^2}{x_{ij}} - \frac{3}{x_{ij}^3} \tag{B.16}$$

$$\sum_{\substack{k=1\\i\neq j\neq k\neq i}}^{N} \frac{1}{x_{ik} x_{kj}^3} = \frac{1}{3} \frac{2N+1-x_j^2}{x_{ij}^2} - \frac{1}{2} \frac{x_j}{x_{ij}} - \frac{4}{x_{ij}^4} \tag{B.17}$$

We now notice that (B.9) is identical to (4.6), the condition for the FP model to be $Y[su(n)]$-symmetric. This proves that the lattice sites of the FP model with N spins are actually the zeroes of the $H_N(x)$ polynomial.

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